Economics 700
Problem Set # 2
Coordination Failures: Property Rights and the Rules of the Game

I. Invidious Consumption

Consider two people, denoted by upper case and lower case letters, who are part of the same society, and whose consumption levels affect their own and each other's utility in the following ways: (1) each derives positive (marginal) utility from their own consumption, (2) negative (marginal) utility from the consumption of the other ("invidious" means "concerning envy"), (3) negative marginal utility is associated with hours of work, (4) and the more the one consumes the greater for the other is the marginal utility of (his or her own) consumption. Let c and C be the consumption levels, h and H be the hours of work expressed as a fraction of the day, u and U be the levels of utility enjoyed by each. You may assume that each can survive if necesssary without with zero consumption, and zero utility. ?

Please answer all the numbered questions below.

1 Restricting the utility function. Suppose the utility function of the first (lower case) person is

$$u = a(c - bC) + gcC + dh^2$$

where a, b, g, and d are constants. You may assume that the other person has an analogous utility function (in other words A = a, b = B, g = G and d = D).

1.1 For the first person, write down all of the derivatives of the utility function mentioned in the above paragraph and indicate what sign each must have to conform to the above description.

1.2 Indicate the range of values of the constants in the utility function which are consistent with these restrictions.

1.3 Imagine a person for whom b = 1. How would you describe this person? Imagine a person for whom b = -1. How would you describe this person?

2 The Invidious Consumption Game. For simplicity we assume that consumption is available only by working, with an hour of work making possible a unit of consumption (there is no saving and no exchange) so c = h and C = H. Suppose the available action set is such that each may choose to work either 8 hrs a day or 6 hours a day (a third of a day or a fourth of a day, the units in which h and H are measured) and a = A = 1, g = G = 1, and d = D = -2 and b = B = 0.5, and that they make this choice non cooperatively, and play the game only once.

2.1 Define what it means to say the two make their choices non-cooperatively.

2.2 Write down the relevant pay off matrix.

2.3 What kind of game is it.

2.4 Circle any Nash equilibria in the payoff matrix.

2.5 Draw a square around any Pareto optima in the payoff matrix.

2.6 Does this game give rise to any coordination failures (define coordination failure and indicate if one occurs in this game.)

2.7 If your answer to the previous question is "yes" explain why coordination failures occur in this problem, and if "no" explain why they do not.

3 Best response functions. Now assume the action set includes all positive values of h and H from zero to 1. Each individual faces a single choice -- how much to consume, or what is equivalent, how many hours to work, and makes this decision non cooperatively.

3.1 What optimizing problem must the first person solve to derive her best response function? (write it down)

3.2 Derive the best response function of the two individuals (let the brf be defined for h and H, though c nd C would have been equivalent, of course.)

3.3 Explain in words the meaning of the brf you have derived: what familiar concepts are being equated?

3.4 What is the effect of the "other" working more hours on the number of hours chosen by the first? Give a precise expression for this effect and determine its sign if you can.

3.5 Indicate what you consider to be reasonable rules (one for each individual) for how they would change the level of their variable (h and H, respectively) when it is not at its equilibrium (i.e. best response) level (i.e. give a precise mathematical statement of the out of equilibrium adjustment processes.)

4 Nash Equilibrium

4.1 Graph the two above brf's and indicate the Nash equilibrium if one exists.

4.2 If a = A = 1, b= B = .5, g = G = 1, and d = D = -2, give the Nash equilibrium values of h and H (label these h* and H*).

4.3 If the two individuals described above were interacting as described in the opening paragraph, what reasons might given to expect them to be performing the Nash equilibrium levels of work.

4.4 Suppose a = A = b = B = -.5, g = G = 5, and as before d = -2. = D
Identify any Nash equilibria. On the basis of the behaviors thus far specified, what levels of work by the two individuals would you expect to observe, or is there no basis for saying anything about this?

5 Pareto optimality.

Stackelberg or First Mover Advant.

5.1 What optimizing problem (for the above interaction) must be solved to derive the marginal conditions defining a Pareto optimum? (Write down the optimizing problem.)

5.2 What are these marginal conditions? (solve the problem)

5.3 Using familiar economic concepts (not simply formal mathematical concepts) explain what these conditions mean.

5.4 Can you show that the Nash values (h* and H*) are not Pareto optimal? Give a precise demonstration of your answer.

6 Cooperative Solutions. Imagine that the two realized that they could both do better by acting cooperatively, and agreed to share equally in the benefits of cooperation (you can assume that any agreement they make is enforceable).

6.1 What optimizing problem would they solve to determine how hard each should work?

6.2 With the values $a = A = 1$, $b = B = .5$, $g = G = 1$ and $d = D = -2$, how much would each work in the cooperative solution?

6.3 Explain why these values, call them h' and H', differ from H^* and h^*.

7 Governmental solutions. Now assume that the two cannot cooperate in determining h and H but that they can agree to instruct the state to tax or subsidize their various activities (consumption and work), the taxes and subsidies being collected and distributed costlessly, the government having full knowledge of each person's levels of work and consumption. Any tax which is collected is distributed in a lump sum (equally) to the two people. You can assume that neither of the two know that the lump sum payment they receive from the state will vary with the amount they work (they consider the lump sum to be exogenous.)

7.1 Considering only taxes, what might the government tax in order to move the Nash equilibrium closer to the values H', h'?

7.2 Does there exist a tax which would induce each person to implement the cooperative outcome (even if each were acting non cooperatively)? If so say what it is, if not explain why it is not possible.

II. Love thy neighbor? The owners of two adjacent buildings of rental apartments (i and j) decide how much labor (denoted e, below) to devote to maintaining the physical appearance of their properties. 1 Increased labor by one raises rental income of his own building (people are willing to pay more for a beautiful yard) and 2 (because it improves the appearance of the entire neighborhood) raises the rental income of the other property. 3 Increased labor effort also raises the marginal revenue productivity of the labor effort of the neighboring landlord for any given level of other inputs.

The rental income net of all costs other than the owner's labor for the property held by i is given by

$$y^i = a + be^i + ce^ie^j$$

and landlord j's net rental income is determined by a perfectly analogous function (simply interchange the subscripts).

The utility functions of the landlords reflect a positive marginal utility of income and a disutility of labor effort. Thus

$$u^i = y^i - g(e^i)^2$$

and analogously for the other landlord. For both landlords, $0 \le e \le 1$ (when $e = 1$ the effort level is at physical maximum).

1 Setting up the problem:

1.1 What mathematical restrictions does the verbal description of the interrelationship among the two landlords impose on the values of a, b, c, and g?

1.2 If each landlord optimizes utility (non cooperatively) what optimizing problem must they solve?

2 Best Responses:

2.1 Derive the best response functions of each landlord. Define your

BRFs as $e_i{}^* = e_i(e_j)$ and analogously for j. (It may help in what follows to graph the functions.)

2.2 Imagine a change in i's utility function reflecting an increase in the disutility of labor effort. What is its effect on the choice of labor effort by i? (Give a precise expression for this effect.)

2.3 Can you say if an increase in i's disutility of labor effort (as described above) will increase, decrease or leave unchanged the effect of changes in j's choice of effort on i's choice of effort? If possible, give an expression both

2.3.1 for the effect of changes in j's effort levels on i's effort levels and

2.3.2 for the effect of an increase in the disutility of i's labor on the responsiveness of i's choice of effort to j's choice of effort.

3 Nash equilibrium:

3.1 Define a nash equilibrium for this game.

3.2 Derive an expression for the value of both landlords' effort levels in the Nash equilibrium.

3.3 If b = 1, c = .25 and g = 2 what are the values of e^i and e^j in the Nash equilibrium you have just defined?

3.4 (10) Suppose that the two landlords fall in love, and now desiring both to impress and to help the other, they find working on the maintanence of their property less onerous, the marginal disutility of labor effort now falling to a quarter of what it was in the example immediately above. What is the new Nash equilibrium for the otherwise unchanged values of question 3.3 above (and assuming that they continue to play non-cooperatively)?

4 Privatization. Imagine that i owned both properties, and could employ j to work on the second property while i continues to work on the first. The wage i will offer j is just large enough to compensate j for the disutility of whatever labor effort i hires j to perform (i will set the wage so that j will be indifferent between working that amount and not working at all). (We are assuming here that effort can be contracted for, on which, more later.)

4.1 Write down i's optimizing problem.

4.2 Using the first order conditions from the optimizing problem immediately above, give the optimal values of the labor of both i and j for b = 1, c = .25 and g = 2

4.3 Compare your result to the Nash equilibrium (for the same parameters) and explain the difference, if there is one, or the reason why there is no difference.

4.4 Is the privatization solution Pareto optimal? Explain your answer.

4.5 Do you have enough information to say if this solution is Pareto superior to the Nash equilibrium you identified above? (If so, indicate the answer, if not, say why not.)

100/100 Again!

Economics 700 - Economic Theory of Coordination, Cooperation and Conflict
Samuel Bowles - Fall 1996
Problem Set #2: **"Coordination Failures: Property Rights and The Rules of the Game"**
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1. Invidious Consumption

The 4 statements about the two people (Lower, and Upper) are interpreted as follows:

i. "Each derives positive (marginal) utility from their own consumption" =>
Non-satiation: $u_c > 0$; $U_C > 0$

ii. "...negative marginal utility from the consumption of the other.." =>
Envy: $u_C < 0$; $U_c < 0$

iii. Negative marginal utility associated with hours of work," =>
Disutility of work: $u_h < 0$; $U_H < 0$

iv. "...the more the one consumes the greater for the other is the marginal utility of his own consumption..". =>
consumerism: $\partial/\partial c(\partial U/\partial C) > 0$; $\partial/\partial C(\partial u/\partial c) > 0$

Let utility function for Lower be: $\mathbf{u = a(c - bC) + gcC + dh^2}$, a,b,g,d constants. (Analogous function for Upper, where a=A, b=B, g=G,d=D)

1.1 Derivatives for "lower":

$\partial u/\partial c = u_c = a + gC > 0$ (from I) - Marginal Utility of own Consumption

$\partial u/\partial C = u_C = -ab + gc < 0$ (from ii) - Marginal Disutility from other's Consumption

$\partial u/\partial h = u_c = 2dh < 0$ (from iii) - Marginal Disutility of own effort (work)

$\partial/\partial C(\partial u/\partial c) = g > 0$ (from vi) - Increasing Marginal Utility of own consumption from other's consumption

1.2 Ranges of values:

Given the interpretations above, the following ranges can be derived for the parameters:

It has been shown that **$g>0$**

From iii) $2dh<0$, since $h>0$ (nonnegativity of labor), => **$d<0$**
From I) $a+gC>0$, and since $C>0$ and $g>0$ => **$a < gC$**
From ii) $-ab + gc < 0$, since $g>0$, $c>0$, => **$ab > gc$, or, $b> gc/a$**

1.3 b=1; b=-1

If $b=1$ => $u=a(c-C)+gcC+dh^2$
In this case, **"lower" acts as an envious person** because of the term (c-C) that identifies the externality from the (envy) disutility of the other's consumption, more precisely, because Upper's consumption C decreases lower's utility.

If $b=-1$ => $u=a(c+C)+gcC+dh^2$
Here, **"lower" acts as an altruist person** given that for the values assumed for the parameters, any increase in Upper's consumption C will increase Lower's utility.

2. Invidious Consumption Game

It is assumed that consumption is possible only through work, and h=c, H=C. Also assumed that a=A=1; g=G=1; d=D=-2; b=B=0.5
Non-cooperative, one shot game
Action set: h={⅓, ¼} (fraction of the day dedicated to work by lower)
H={⅓, ¼} (fraction of the day dedicated to work by upper)

2.1 Definition of a Non-cooperative game: Such game assumes that the players cannot write any contracts or make any binding agreements before they take their respective decisions from the action set of the game.

2.2 The payoffs matrix is derived from estimating the utility levels for Lower and Upper for each of the possibilities of the action set, h=⅓ and h=¼

Payoff matrix:		Upper	
		H=⅓	H=¼
Lower	h=⅓	4/72 ; 4/72 Nash	5/72 ; 3/72 P.O.
	h=¼	3/72 ; 5/72 P.O.	4.5/72 ; 4.5/72 P.O.

2.3 Given the characteristics of the game, it is a **PRISONERS DILEMMA game**, for the following features of the payoffs matrix.

2.4 There is one Nash equilibrium which is Pareto inferior, as shown in the matrix.

2.5 All outcomes different from the Nash equilibrium are Pareto optimal (see matrix)

Besides, such game and payoffs has a dominant strategy, namely, lower should work h=⅓ of a day, and upper should work H=⅓ of a day.

2.6 A Coordination Failure arises when there are non-cooperative interactions in which players do not take appropriate account of the effects of their actions on the well-being of others and there is no Pareto optimal solution.

This game does include a coordination failure basically because the utility functions of the individuals are sharing variables, meaning, there are externalities derived from each other's behavior that are not being internalized and therefore they are not taking account of the effects of their actions. Given the structure of u and U, the level of consumption of Lower and Upper will affect each other's level of utility.

A proof that the outcome of the game includes a coordination failure is that the Nash solution is non Pareto optimal. In other words, if both decided to work ¼ of a day, the payoffs for both will be higher (4.5/72).

2.7 There are three possible causes of coordination failures, that the Pareto Inferior solution is a stable Nash equilibrium, that a No Pareto Optimal outcome is a stable Nash equilibrium or that there is not Nash equilibrium or ways to play the game. In this case, the two first reasons occur for this game and a re typical of Prisoners Dilemma games.

3 Best Response Functions:

Action set: $\{h \in (0,1),\ H \in (0,1)\}$

3.1. Optimizing problem:

$$\underset{(h)}{\text{Max}} \left\{ u = a(h - bH) + ghH + dh^2 \right\} \quad , g>0,\ d<0,\ c, C>0$$

$(h = c)$

3.2 BRF: $h(H)$: $\frac{\partial u}{\partial h} = a + gH + 2dh = 0$ (FOC)

(lower) $\Rightarrow \boxed{h(H) = \frac{-1}{2d}(a + gH)} = \frac{-a}{2d} - \frac{g}{2d}H$

(upper) $\boxed{H(h) = \frac{-1}{2D}(A + Gh)}$ (by symmetry) $H(h) = \frac{-1}{2d}(a + gh)$

3.3 The FOCs say $\underbrace{a + gH}_{u_c} + \underbrace{2dh}_{-u_h} = 0$

Meaning that "lower" is equating her marginal utility of consumption to her marginal disutility of work. In the case of "upper" the interpretation is exactly the same.

3.4 As we see in the FOCs, the brf. depends on the level of work the other persons chooses.

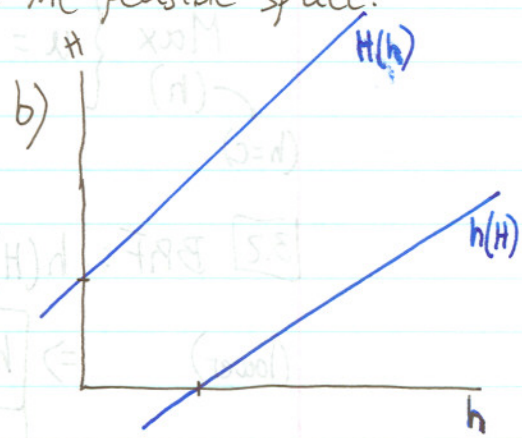
Thus, $\frac{\partial h(H)}{\partial H} = \frac{-1}{2d} \cdot g = \boxed{-g/2d = h_H}$ since $d<0$ and $g>0$,

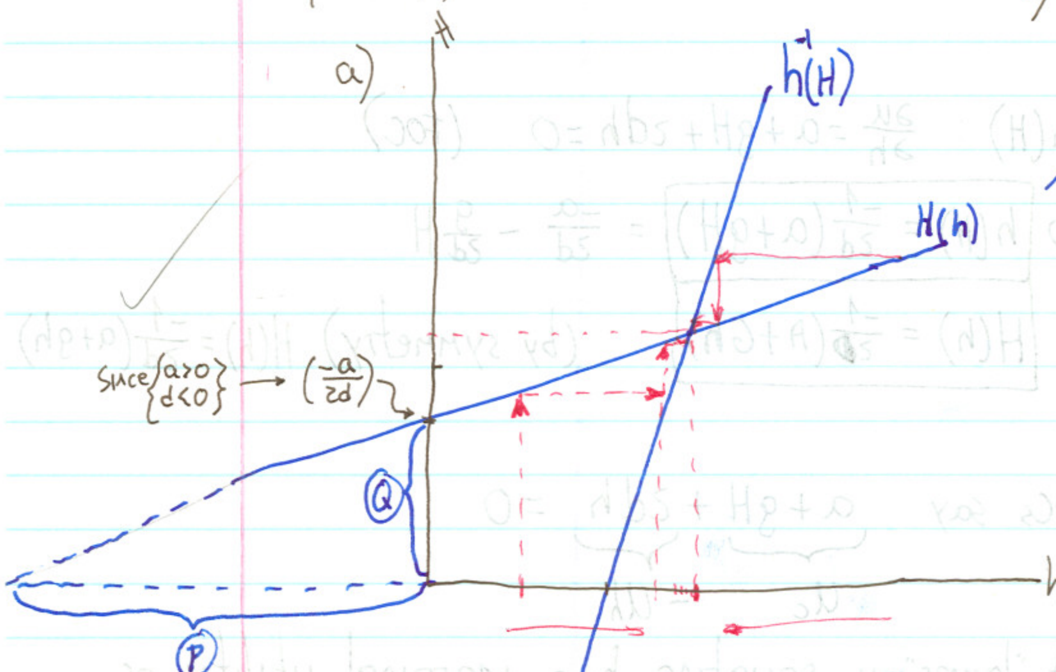
$\Rightarrow h_H > 0$, meaning that the more one person consumes, the higher is the best reaction of the other by $(-g/2d)$ times.

$\boxed{3.5}$ Out of equilibrium adjustment:

From $\boxed{3.4}$ we know that the brf. are increasing (upward). In such case two options can happen, a) that the two brf find an equilibrium at certain point where $h>0$, $H>0$, or b) that no equilibrium exists in the feasible space.

Graphically,





$h(H) = \frac{-a}{2d} - \frac{g}{2d}(H)$

$H(h) = \frac{-a}{2d} - \frac{g}{2d}(h)$

Our actual model behaves as in a) which can be proved by showing that the angle of $H(h)$ is less than 45°, i.e. that P > Q in the picture. Q $= (-a/2d)$ intercept of $H(h)$, and P is found by setting $H(h) = 0 = -a/2d - g/2d\, h$ and solving for h.

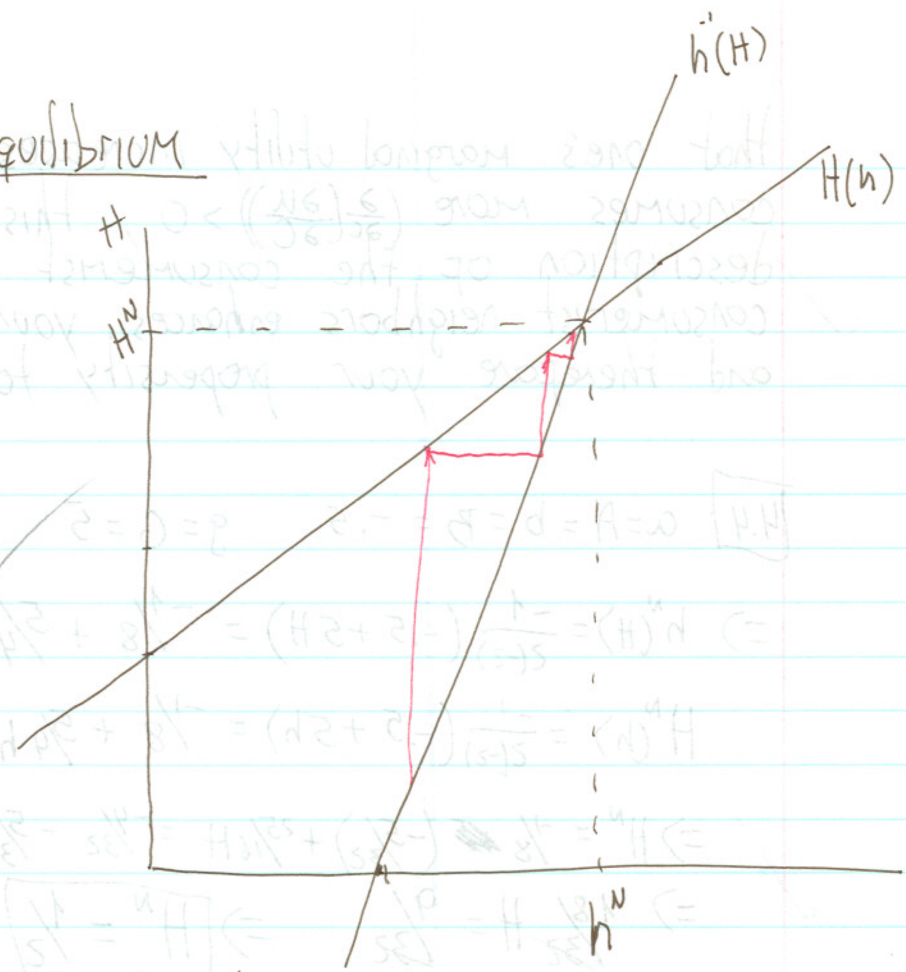
Since the slope of $h^{-1}(H)$ is higher than slope of $H(h)$, any adjustment process takes any h, H levels towards equilibrium (i.e. intersection) at a decreasing rate. In other words the ~~[illegible]~~ reaction to one's decision is always of equal sign but lower size

Mathematically, since $dh/dH < 1$, $dH/dh < 1$ (<=> slopes are < 45%)

$\Rightarrow \left(\frac{dh}{dH}\right)\left(\frac{dH}{dh}\right) < 1$ (product of 2 fractions is a fraction)

4 Nash Equilibrium

4.1 Once again,



The Nash Equl. exists at (H^N, h^N)

4.2 $a = A = 1$; $b = B = 1/2$; $g = G = 1$; $d = D = -2$

$$\Rightarrow h^*(H) = 1/4 + H/4 \;;\; H^*(h) = 1/4 + h^*/4$$

$$\Rightarrow h^*(H) = 1/4 + 1/4\left(1/4 + h^*/4\right) = 1/4 + 1/16 + 1/16\,(h^*)$$

$$\Leftrightarrow \left(\frac{16-1}{16}\right) h^* = 1/4 + 1/16 = 5/16 \Leftrightarrow \boxed{h^* = 1/3} \Rightarrow \boxed{H^* = 1/3}$$

Notice it's the same outcome that the P.D game predicted from the payoffs matrix.

4.3 The explanation that the individuals' best reaction is to work more, until $h = H = 1/3$ day lies on the definition of their util. functions and the behavior described. Since their marginal utilities of consumption are positive, and $c = h$, and the "envy" externality plus the second-derivative property that one's marginal utility increases as the other consumes more $\left(\frac{\partial}{\partial c}\left(\frac{\partial u}{\partial C}\right)\right) > 0$, this could be a good description of the consumerist society: living around consumerist neighbors enhances your propensity to consume and therefore your propensity to work more.

$\boxed{4.4}$ $a = A = b = B = -.5 \quad g = G = 5 \quad d = -2 = D$

$$\Rightarrow h^N(H) = \frac{-1}{2(-2)}(-.5 + 5H) = -1/8 + 5/4\, H$$

$$H^N(h) = \frac{-1}{2(-2)}(-.5 + 5h) = -1/8 + 5/4\, h = -1/8 + 5/4(-1/8 + 5/4\, H)$$

$$\Rightarrow H^N = -1/8 \left(-5/32\right) + 25/16\, H = -4/32 - 5/32 + 50/32\, H$$

$$\Rightarrow 18/32\, H = 9/32 \quad \Rightarrow \boxed{H^N = 1/2}\,;\ \boxed{h^N = 1/2}$$

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5 Pareto Optimality:

5.1 To find a pareto optimal solution, we now need to optimize the utility of one person without decreasing the utility of the other.

Thus, $\max_{(h,H,\lambda)} \{ u = u(h,H) \text{ s. to } U = \underline{U} \}$

Lagrangian, $\mathcal{L} = a(h-bH) + ghH + dh^2 - \lambda[\underline{U} - a(H-bh) + gHh + dH^2]$

5.2 FOCs. ① $\frac{\partial \mathcal{L}}{\partial h} = a + gH + 2dh + \lambda ab - \lambda gH = 0$

② $\frac{\partial \mathcal{L}}{\partial H} = -ab + gh - \lambda a - \lambda gh - 2\lambda dH = 0$

③ $\frac{\partial \mathcal{L}}{\partial \lambda} = a(H-bh) + gHh + dH - \underline{U} = 0$

$\Rightarrow$ From ① $\boxed{h^{po} = -\frac{1}{2d}[-ab + gH + \lambda(a+gh)]}$

Notice how the marginal conditions differ due to the constrained maximization, in the definition of the brf.

From ② and by symmetry $H^{po} = -\frac{1}{2d}[\underbrace{-ab + gh}_{} + \lambda\underbrace{(a+gh)}_{}]$

5.3 (U_c) (u_C) (envy) Marginal Util. from others' consumption $+ \lambda$ [Marginal Utility of Consumption]

The first order conditions are saying that now "lower" is equating her marginal disutility from others consumption to the marginal utility of consumption, corrected by λ, the "shadow price" of the constraint condition.

$\boxed{5.4}$ To show that the Nash solution is Pareto Inferior to the P.Optimal, we need to prove that somebody is worse off when P.O $\longrightarrow$ Nash.

Nash: $h^N = \frac{-1}{2d}(a + gH)$

$$h^{OPT} = \frac{-1}{2d}\left[(a+gH) + \lambda\underbrace{(-ab+gH)}_{\ominus}\right]$$

Since $(-ab + gH) < 0$ $\left((\Rightarrow) \partial u/\partial c < 0 \text{ (envy)}\right)$, and $\lambda > 0$

$\Rightarrow$ The difference between h^N and h^{OPT} is negative. i.e. $\underline{h^* > h^{OPT}}$

Since $\partial u/\partial h < 0$ (disutility of work)

$\Rightarrow U^* < U^{OPT}$, i.e. Nash is inferior, since "lower's" utility is decreased, even if "Upper's" utility were constant or better.

6 Cooperative Solution

Acting cooperatively means that the two individuals coordinately maximize their joint utility functions and distribute equally the gains.

6.1 $$\max_{(h,H)} U_{Total} = \left\{ \left[a(h-bH) + ghH + dh^2\right] + \left[a(H-bh) + gHh + dH^2\right] \right\}$$

FOCs, $\frac{\partial UT}{\partial h} = a + gH + 2dh - ab + gH = 0$

$\frac{\partial UT}{\partial H} = -ab + gh + a + gh + 2dH = 0$

From which, $h^{coop}(H)$, $H^{coop}(h)$ can be solved. $h^{coop} = \frac{ab-a}{2(g+d)}$

6.2 $a = A = 1$; $b = B = 1/2$; $g = G = 1$; $d = D = -2$

$UT = 1(h - \frac{1}{2}H) + hH - 2h^2 + H - \frac{1}{2}h + hH - 2H^2$

$UT = \frac{1}{2}h - 2h^2 + \frac{1}{2}H - 2H^2 + 2hH$

$\partial UT/\partial h = \frac{1}{2} - 4h + 2H = 0 \iff h = \frac{1}{4}(\frac{1}{2} + 2H) = \frac{1}{8} + \frac{1}{2}H$

$\partial UT/\partial H = \frac{1}{2} - 4H + 2h = 0 \iff \frac{1}{2} - 4H + 2[\frac{1}{8} + \frac{1}{2}H] = 0$

$\iff \frac{1}{2} + \frac{2}{8} = 4H - H$

$\iff \boxed{H^{coop} = \frac{1}{4}} = H'$

$\boxed{h^{coop} = \frac{1}{4}} = h'$

6.3 In the coop. solution, the coordination failure has been solved by internalizing the externality from (envy) negative marginal utility from other's consumption. The game has changed, it's a coop. game now, therefore, they can work less and not waste energy in the consumpt[illegible]

7 Government Intervention:

A tax on behavior (consumption) whose revenues are collected and distributed back as lump sums and exogeously for "lower" or "upper".

7.1 Since the origin of the Coordination Failure is on the interdependence on consumption, the tax should be imposed on consumption c, C.

The new situation is then,

$$u = a(c - bC) + gcC + dh^2 + \underbrace{\tau}_{\text{Tax}} h$$

To find τ, we can look at the difference between the optimal and Nash solutions' first order conditions:

(optimal) $\partial u/\partial h = a - ab + 2gh + 2dh = 0$

(Nash) $\partial u/\partial h = a + gH + 2dh = 0$

difference: $[-ab + (-gH) + 2gh]$ Since symmetry of utilities and FOCs $\Rightarrow h = H$

$\Rightarrow$ FOCs difference $= gH - ab = (gh - ab)$

This is precisely the value of the new tax on consumption that will yield an optimal solution equivalent to the cooperative solution, since such tax induces each individual to take account of her actions through an incentive (negative) on her consumption.

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II $\boxed{\text{Love Thy Neighbor}}$

2 owners of 2 adjacent buildings (i), (j)

i) Marginal Income of labor: $\partial Y^i/\partial e^i > 0$; $\partial Y^j/\partial e^j > 0$

ii) [Externality] from neighborhood: $\partial Y^i/\partial e^j > 0$; $\partial Y^j/\partial e^i > 0$

iii) Increasing Marginal (Y) productivity of neighbor as e of other ↑:

$$\frac{\partial}{\partial e^i}\left(\frac{\partial Y^j}{\partial e^j}\right) > 0 \; ; \; \frac{\partial}{\partial e^j}\left(\frac{\partial Y^i}{\partial e^i}\right) > 0$$

$\boxed{1.1}$ $U_i = Y_i - g e_i^2$, $\partial U_i/\partial e_i < 0$ (Disutility of effort)

$Y_i = a + be^i + ce^ie^j$ (Income with externality) $\boxed{g > 0}$

$\Rightarrow U_i = [a + be^i + ce^ie^j] - ge_i^2$ $\quad 0 \le e \le 1$

$\boxed{U_i = a + be^i + ce^ie^j - ge^{i^2}}$ $= V(e_i, e_j)$; $dV = \frac{\partial}{\partial e_i}(U)de_i + \frac{\partial}{\partial e_j}(\,)de_j = 0$

From i) $\partial Y_i/\partial e^i = b + ce^j \; \cancel{} > 0$ (Marginal Utility (Income) of labor)

ii) $\partial Y_i/\partial e^j = ce^i > 0 \Rightarrow \boxed{c > 0}$

iii) $\frac{\partial}{\partial e^j}(\partial Y^i/\partial e^i) = \boxed{c > 0}$

$\Rightarrow b > 0$ or

$b < 0 \Leftrightarrow |b| < ce^j$

$\boxed{ce^j > -b}$

$\boxed{1.2}$ Optimizing problem: (Non-coop, Nash game)

$$\max_{(e_i)} U_i = a + be_i + ce_ie_j - ge_i^2$$

2. Best Response Functions

$\boxed{}$ Max $u_i = a + be_i + ce_ie_j - ge_i^2$

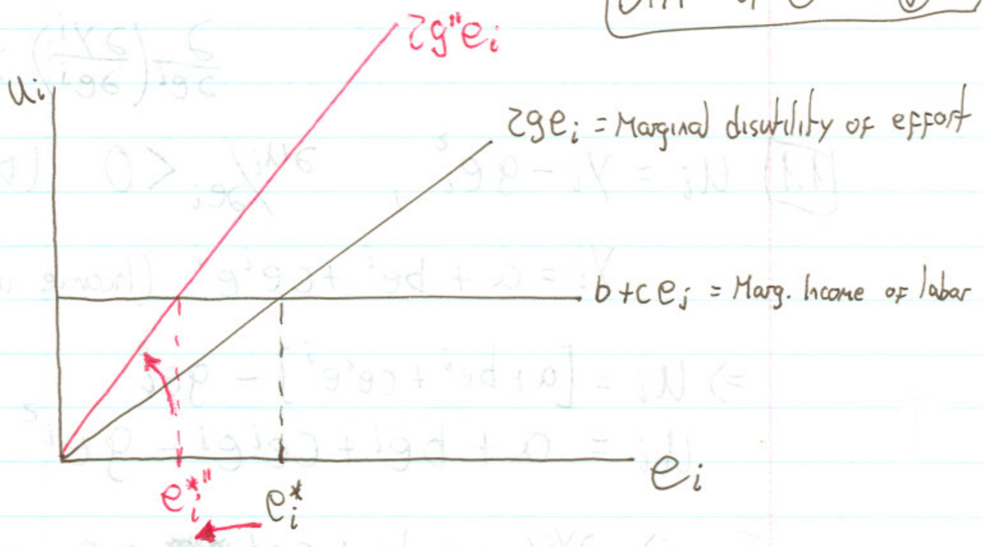
$\boxed{2.1}$ FOCs: $$\partial u_i / \partial e_i = \underbrace{b + ce_j}_{\text{Marginal Utility of labor (from income)}} - \underbrace{2ge_i}_{\text{Marginal Disutility of effort}} = 0 \Rightarrow \boxed{e_i^* = \frac{1}{2g}(b + ce_j)}$$

$$\boxed{e_j^* = \frac{1}{2g}(b + ce_i)}$$

BRF of i and j

$$\partial u_i / \partial e_j = ce_i = 0$$

Graphically,



$\boxed{2.2}$ An increase in the disutility of labor effort could be interpreted as an increase in the "$2ge_i$" component, mainly by a change in its slope g. (see new red marg. disutility of effort).

Such change induces a decrease in the level of optimal effort e_i^* by i (see picture) in order to maintain optimality.

$$\partial e_i^* / \partial g = \partial / \partial g \left(\tfrac{1}{2}(b + ce_j) g^{-1} \right) = -\tfrac{1}{2}(b + ce_j) g^{-2} < 0$$

$\boxed{2.3}$ Again, $e_i^* = \frac{1}{2g}(b + ce_j)$; $e_j^* = \frac{1}{2g}(b + ce_i)$

Disutility of labor (effort): $\partial u_i / \partial e_i = -2ge_i$

$\boxed{2.3.1}$ $\frac{\partial e_i^*}{\partial e_j} = \frac{c}{2g}$, since $c>0$, $g>0$ $\Rightarrow \frac{\partial e_i^*}{\partial e_j} > 0$

Thus, each will work harder ($\uparrow e$) as a best response to the other's increase in effort.

$\boxed{2.3.2}$ $\frac{\partial}{\partial g}\left(\frac{\partial e_i^*}{\partial e_j}\right) = -\frac{1}{2}cg^{-2} = \frac{-1}{2g^2} c^{\oplus} < 0$

But each will decrease effort as the other's disutility of labor increases.

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3 **Nash Equilibrium**: Each individual optimizes her effort level assuming the other's (e) ~~expected~~ best reaction.

$\boxed{3.1}$

Thus, $\text{Max } u_i = a + be_i + ce_ie_j - ge_i^2$

$\boxed{3.2}$ and when solving $e_i^* = e_i^*(e_j) = e_i^*(e_j(e_i))$ i.e. Nash behavior.

$$\Rightarrow e_i^* = \frac{1}{2g}\left(b + c\left[\tfrac{1}{2g}(b + ce_i)\right]\right) =$$

$$= \frac{b}{2g} + \frac{c}{2g}\left[\frac{b}{2g} + \frac{ce_i}{2g}\right] = \frac{b}{2g} + \frac{cb}{4g^2} + \frac{c^2e_i}{4g^2}$$

$$\Rightarrow e_i\left(1 - \frac{c^2}{4g^2}\right) = \frac{b}{2g} + \frac{cb}{4g^2} \Rightarrow e_i^N = \frac{\frac{b}{2g} + \frac{cb}{4g^2}}{1 - \frac{c^2}{4g^2}} = \frac{\frac{b}{2g}\left(1 + \frac{c}{2g}\right)}{\left(\frac{4g^2}{4g^2} - \frac{c^2}{4g^2}\right)}$$

$$\Rightarrow e_i^N = \frac{2gb\left(1 + \frac{c}{2g}\right)}{4g^2 - c^2} = \frac{2gb + bc}{(2g-c)(2g+c)} = \frac{b\cancel{(2g+c)}}{(-)\cancel{(+)}}$$

$$\Rightarrow \boxed{e_i^N = \frac{b}{2g-c}} \Rightarrow \boxed{e_j^N = \frac{b}{2g-c}} \text{ (symmetry}$$

3.3 $b=1,\ c=\frac{1}{4},\ g=2$

$$\Rightarrow e_i^N = \frac{b}{2g-c} = \frac{1}{2(2)-\frac{1}{4}} = \frac{1}{\frac{16}{4}-\frac{1}{4}} = \frac{1}{\frac{15}{4}} = \boxed{\frac{4}{15} = e_i^N = e_j^N}$$

3.4 Love:

New marginal disutility of labor effort $= \frac{1}{4}[-2ge_i] = -\frac{1}{2}ge_i$

New FOCs: $b+ce_j = \frac{1}{2}ge_i \Rightarrow \boxed{e_i^{love} = \frac{2(b+ce_j)}{g}}$ BRF (i)

$$\boxed{e_j^{love} = \frac{2(b+ce_i)}{g}} \text{ BRF (j)}$$

Thus, $e_i^{love} = e_i^{love}(e_j^{love}(e_i))$

$$= \frac{2b}{g} + \frac{2ce_j}{g} = \frac{2b}{g} + \frac{2c}{g}\left[\frac{2b}{g} + \frac{2ce_i}{g}\right]$$

$$e_i^L = \frac{2b}{g} + \frac{2c}{g}\left(\frac{2b+2ce_i}{g}\right) = \frac{2b}{g} + \frac{4cb}{g^2} + \frac{4c^2e_i}{g^2}$$

$$\Rightarrow e_i\left(1-\frac{4c^2}{g^2}\right) = \frac{2b}{g}\left(1+\frac{2c}{g}\right) \Rightarrow \boxed{e_i^{love} = \frac{\frac{2b}{g}\left(1+\frac{2c}{g}\right)}{\left(1-\frac{4c^2}{g^2}\right)}}$$

Substituting $b=1,\ c=\frac{1}{4},\ g=2$

$$\Rightarrow e_i^{love} = \frac{\frac{2}{2}\left(1+\frac{2/4}{2}\right)}{\left(1-\frac{4/16}{4}\right)} = \frac{4}{3}$$

! For love they would work 32 hours a day!! (I would too!)

Since $0 \le e \le 1 \Rightarrow$ $\boxed{\text{Nash (love) solution is } e_i = e_j = 1}$

—o—

4 Privatization:

Now i owns both properties (i.e. gets all income) and employs j for wage = former disutility of effort.

New Max. problem: i derives utility from both sources of income. Although she is still working in one building, now she has to pay j, a wage equivalent to his former disutility of effort. In other words, her problem ressembles a joint utility funtion where:

$$U_i = \underbrace{[a + be_i + ce_ie_j] + [a + be_j + ce_je_i]}_{\text{[i's Income from both buildings]}} \underbrace{-ge_i^2}_{\text{Disutility of i's effort}} \underbrace{-ge_j^2}_{\text{Wage to j.}}$$

4.1 $\max_{(e_i, e_j)} U_i = 2a + 2ce_ie_j + be_i + be_j - ge_i^2 - ge_j^2$

4.2 $\partial U_i / \partial e_i = 2ce_j + b - 2ge_i = 0 \Rightarrow \boxed{e_i^* = \frac{1}{2g}(2ce_j + b)}$

$\partial U_i / \partial e_j = 2ce_i + b - 2ge_j = 0 \Rightarrow \boxed{e_j^* = \frac{1}{2g}(2ce_i + b)}$

Given that the income productivity of labor for both is the same,

$\Rightarrow e_i^* = \frac{1}{2g}(2ce_i + b) \Rightarrow 2ge_i^* = 2ce_i^* + b$

$\Rightarrow \boxed{e_i^* = \frac{b}{2(g-c)} = e_j^*}$

If $b = 1$, $c = 0.25$, $g = 2$

$\Rightarrow \boxed{e_i = e_j = \frac{1}{2(2-.25)} = 2/7}$

(11)

4.3 $e_i^N = e_j^N = 4/15 = 0.267$ of a day (Nash solution)

$e_i^* = e_j^P = 2/7 = 0.286$ of a day (Private sol.)

Under the private game, the joint utility ~~[illegible]~~ maximization is correcting the coordination failure derived from the positive externality on the income functions, since (i) is now controlling both effort levels.

Thus, (i) is getting more utility under the private solution:

$$u_i^{Nash} = a + be_i + ce_ie_j - ge_i^2 = \boxed{a + (4/15)} + \cancel{1/4 (4/15)^2} - \cancel{(1/4)(4/15)^2}$$

$$u_i^{private} = 2[a + be_i + ce_ie_j] - ge_i^2 - ge_j^2$$

$$= 2[a + (2/7) + (1/4)(2/7)^2] - 2(2/7)^2 - 2(2/7)^2$$

$$= 2a + 4/7 + 1/2(4/49) - 4(4/49) = 2a + 28/49 + (1/2 - 4)(4/49)$$

$$= \boxed{2a + (2/7)}$$

Such optimization, given the specific dimensions and signs of their marginal utilities from effort, income, etc. yield a solution where they work a little harder and improve ~~[illegible]~~ (i)'s level of wellbeing while maintaining (j)'s ~~[illegible]~~ disutility of work compensated through wage

4.4 Yes, it is Pareto Optimal since any change in e_i or e_j would make somebody worse off. If (i) works less, her income will decrease, if (j) work less his utility remains same but (i) (owner) will see her income decrease.

$\boxed{4.5}$ In $\boxed{4.3}$ we compared the utilities levels for the Nash and Private solutions. As shown above, for the same parameters, $U_i^{Nash} = a + 4/15$

$$U_i^{Priv} = 2a + 2/7$$

Unless a takes negative values, $U^{Private} > U^{Nash}$

Thus, the private solution is pareto superior from i's stand point.

However, from j's pointview, he used to derive certain gains from owning his building which now he doesn't, since the wage he earns basically covers his disutility of effort, he is surely worse off now from such expropriation process.

Thus, the private solution is NOT pareto superior under the new game.